Bosonization of QED fermions on the (3+1)D lattice

2004 J. Phys. A: Math. Gen. 37 8901

(http://iopscience.iop.org/0305-4470/37/37/010)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 8901–8909 PII: S0305-4470(04)80392-5

Bosonization of QED fermions on the (3+1)D lattice

Adam Bednorz

Institute of Theoretical Physics, Warsaw University, ul. Hoza 69, PL00-681 Warsaw, Poland

E-mail: abednorz@fuw.edu.pl

Received 6 May 2004
Published 1 September 2004
Online at stacks.iop.org/JPhysA/37/8901
doi:10.1088/0305-4470/37/37/010

Abstract

The fermionic quantum electrodynamical determinant is expressed by a bosonic partition function for configurations of worldsheets. The Pauli minus sign of fermionic loops is obtained from the local structure of a worldsheet. The action is local and gauge independent. The 1–1 correspondence is discussed and proved in an asymptotic limit.

PACS number: 03.65.Ca

1. Introduction

Nature, according to every version of *quantum field theory*, consists of two kinds of particles: bosons and fermions. It is clear when writing the fundamental partition function for fields [1], which is the basis of all quantum-mechanical computations. Bosonic fields, namely, are expressed by commuting functions of spacetime while fermionic fields need anticommuting Grassmann variables. We claim that it is possible to get rid of Grassmann variables and write the fermionic partition function only in terms of bosonic fields.

Such a procedure is called bosonization and has quite a long history. Since Wilson [2] it has been known that the configuration of a pair of fermions (fermion and antifermion) can be expressed by a (Wilson) line in 3D or surface in (3+1)D to solve the gauge dependence problem in quantum electrodynamics (QED). Because of fundamental problems in continuum theory, that idea has been rather developed on the lattice both for QED and chromodynamics [3–5]. Before accomplishing that task, some problems like fermion doubling had arisen [5]. Moreover, most of the literature about lattice QED neglects the problem of bosonization. Several papers that nevertheless marked huge progress in this field [6–8] did not mention two important issues: the 1–1 correspondence between bosonic and fermionic partition functions and the origin of the Pauli minus sign for loops which define the boundary of a worldsheet. The minus should appear as a result of local operations on the interior of the worldsheet, not just the topology of the boundary.

In this paper, we address these two points: the minus and 1–1 correspondence. We do not, however, forget about other essential difficulties such as fermion doubling. To solve that last problem we apply a standard method—the Wilson term [9, 10], stressing that it does not affect our main results significantly. The gauge field action is omitted here because it does not influence our idea.

The paper is arranged as follows. In section 2 we remember the structure of the vacuum QED fermionic determinant in the lattice case. In section 3 we replace fermionic configurations by bosonic Wilson lines and propose a local action. The problem of 1–1 correspondence is discussed in section 4. Section 5 is devoted to conclusions and the appendix contains a possible method of managing fermion doubling.

2. Fermionic determinant

As we noted in the beginning, the main difference between bosons and fermions is that the description of fermions involves Grassmann variables [1]. Every two Grassmann variables ξ and η satisfy the anticommutation relation $\xi\eta = -\eta\xi$ and integrals are defined

$$\int \mathrm{d}\xi(a + b\xi) = b \tag{1}$$

with $\mathrm{d}\xi\ \mathrm{d}\eta = -\mathrm{d}\eta\,\mathrm{d}\xi$. The fundamental object in *quantum field theory* is the partition function

$$Z_\eta = \int \mathcal{D}A \int \mathcal{D}\eta\ \exp(\mathrm{i}S[A, \eta]), \qquad S = \int \mathrm{d}^4x\, \mathcal{L}[A, \eta, x] \tag{2}$$

where A represents all bosonic (commuting) fields and η all fermionic fields in spacetime and we integrate over all allowed configurations of fields. The Lagrangian $\mathcal{L}[A, \eta, x]$ must be local—depend only on fields close to x. The objective of bosonization is to replace η by a bosonic field B so that the 1–1 correspondence

$$Z_B = \int \mathcal{D}A \int \mathcal{D}B\ \exp(\mathrm{i}S[A, B]) = C Z_\eta \tag{3}$$

with a constant C can be established. One should also consider possible local configurations of B. We shall construct the field B and $S[A, B]$ for QED although it will not look so simple as the above equation. Equation (3) can be rewritten in the lattice case in the form

$$Z_B = \sum_{A,B} \prod_x Q(A, B, x) \tag{4}$$

where Q is a complex function of fields close to x and the sum runs over allowed configurations.

The QED action for fermions of mass m in spacetime $x = (x^0, x^1, x^2, x^3) \equiv (t, x, y, z)$ is defined in terms of four-component Grassmann fields $\psi(x)$ and $\bar{\psi}(x)$

$$S = \int \mathrm{d}^4x\ \bar{\psi}(x)(\mathrm{i}\not{D}_\mu - m)\psi(x), \qquad \not{D} = \sum_{\mu=0}^{3} \gamma^\mu D_\mu \tag{5}$$

where four-dimensional Dirac matrices $\gamma^{0,1,2,3}$ satisfy

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}I \tag{6}$$

with $g^{00} = -g^{11} = -g^{22} = -g^{33} = 1$ and $g^{\mu\nu} = 0$ for $\mu \neq \nu$ and

$$D_\mu = \partial_\mu - \mathrm{i}eA_\mu = \frac{\partial}{\partial x^\mu} - \mathrm{i}eA_\mu \tag{7}$$

where $A_\mu(x)$ is the electromagnetic potential and e is the electric charge. To have properly shifted poles, which is important in physical calculations, one has to redefine $m \to m - \mathrm{i}|\epsilon|$ with $\epsilon \to 0$. The vacuum fermionic partition function can be written in the form of determinant

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \ \exp(\mathrm{i}S) = \det(\mathrm{i}\not{D} - m) \tag{8}$$

where

$$\mathcal{D}\bar{\psi}\mathcal{D}\psi = \prod_x \prod_{a=0}^{3} \mathrm{d}\bar{\psi}_a(x)\,\mathrm{d}\psi_a(x). \tag{9}$$

The quantity in equation (8) has no sense in itself but the true object necessary to conduct calculations in QED is

$$\frac{\det(\mathrm{i}\not{D} - m)}{\det(\mathrm{i}\not{\partial} - m)}. \tag{10}$$

We are interested in a discretized version of that determinant for a (3+1)D finite lattice. Firstly, we fix our choice of Dirac matrices

$$\begin{aligned} &\gamma^0 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \qquad \gamma^k = \begin{bmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{bmatrix}, \\ &\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma^2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \qquad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \end{aligned} \tag{11}$$

The lattice spacing a will be the same in each direction. All possible points have the structure $x = an$, where $n = (n_0, n_1, n_2, n_3)$ is a group of integers. We decompose each four-component spinor in Susskind's way [5]. Namely (figure 1),

$$\psi(x) = \begin{cases} \psi_0(x) & \text{if } n_0 + n_3 \equiv 0 \bmod 2 \quad \text{and} \quad n_1 + n_2 \equiv 0 \bmod 2, \\ \psi_1(x) & \text{if } n_0 + n_3 \equiv 1 \bmod 2 \quad \text{and} \quad n_1 + n_2 \equiv 1 \bmod 2, \\ \psi_2(x) & \text{if } n_0 + n_3 \equiv 1 \bmod 2 \quad \text{and} \quad n_1 + n_2 \equiv 0 \bmod 2, \\ \psi_3(x) & \text{if } n_0 + n_3 \equiv 0 \bmod 2 \quad \text{and} \quad n_1 + n_2 \equiv 1 \bmod 2. \end{cases} \tag{12}$$

The operator D_μ now has the form

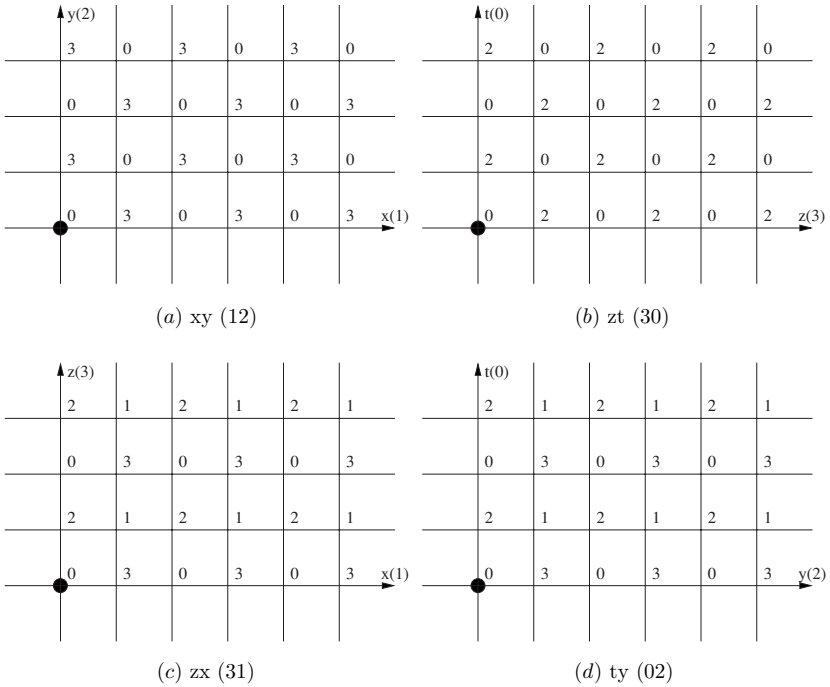
$$(D_\mu\psi)(x) = [\exp(-\mathrm{i}eaA_\mu(x + a_\mu/2)\psi(x + a_\mu)) - \exp(\mathrm{i}eaA_\mu(x - a_\mu/2)\psi(x - a_\mu))]/2a, \tag{13}$$

where $(a_\mu)^\nu = a\delta_\mu^\nu$ and electromagnetic potential A_μ is localized not on lattice sites but on edges with the weight

$$A_\mu(x + a_\mu/2) = a^{-1}\int_x^{x+a_\mu} \mathrm{d}s^\mu A_\mu(s). \tag{14}$$

Due to our special choice of spinor splitting and Dirac matrices we have

$$\not{D}\psi(x) = \begin{cases} (D_0\psi)(x) \\ (-1)^{n_0+n_1+n_2+n_3}(D_1\psi)(x) \\ \mathrm{i}(-1)^{n_0+n_3}(D_2\psi)(x) \\ (-1)^{n_0+n_3}(D_3\psi)(x) \end{cases} \tag{15}$$



(*a*) xy (12) (*b*) zt (30)

(*c*) zx (31) (*d*) ty (02)

Figure 1. Placement of sites in planes. The black point is the point (0, 0, 0, 0).

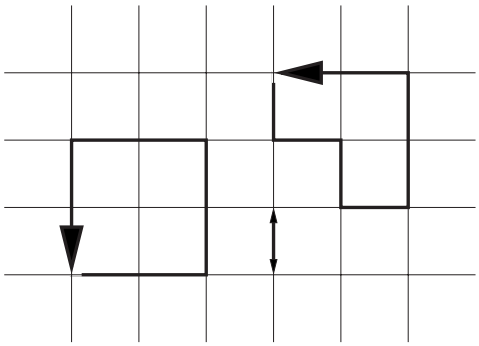


Figure 2. A configuration with two oriented loops and one needle. The loops can be represented by at least three and four plaquettes.

3. Bosonization

The determinant becomes (8) after discretization of the sum

$$\det(\mathrm{i}\not{D} - m) = \sum_C Z_C = \sum_C (-1)^{L(C)} m^{X(C)} \prod_{i \in C} \not{D}(x_{i+1}, x_i) \tag{16}$$

over all possible sets of disjoint oriented closed loops, where $L(C)$ is the number of loops and $X(C)$ is the number of sites not belonging to C. In the degenerate case a needle can occur (figure 2). The product runs over all loops in C and

$$\not{D}(x_{i+1}, x_i) = (\not{D}\delta(x, x_{i+1}))(x_i). \tag{17}$$

Such a contour representation cannot be used for bosonization because $L(C)$ is a non-local object. One has to know the topology of the loops to determine it. In addition, the above covariant derivative is gauge dependent—it needs A_μ instead of its derivatives. The problem can be solved, as we shall present, by introducing plaquette representation [6]. The rough idea is to span a surface on the loop. As shown by Wilson [2], it makes the action gauge independent. Our contribution is showing that the number $L(C)$ can also be locally distributed on such a surface.

We introduce the plaquette p as a square spanning four neighbouring sites with sides oriented either clockwise or counterclockwise and the needle w as a pair of neighbouring sites (a link). Now, we can replace the product (16) by another function of plaquettes in a sheet P spanned on a given loop configuration C and needles, that is

$$\prod_{i\in C}(\gamma^\mu D_\mu)(x_{i+1}, x_i) = \prod_w \eta(w) \prod_l \epsilon(l) \prod_{p\in P}(-1)^R \exp[\mathrm{i}eF_{\mu\nu}(p)] \tag{18}$$

where R is the number of plaquettes 12, 23, 31 and μ, ν are directions of two consecutive sides pointing to increasing x^μ and x^ν, respectively. The field $F_{\mu\nu}$ is assigned to every plaquette with the weight

$$F_{\mu\nu} = a^{-2}\int_{x^\mu}^{x^\mu+a} \mathrm{d}s^\mu \int_{x^\nu}^{x^\nu+a} \mathrm{d}s^\nu(\partial_\mu A_\nu - \partial_\nu A_\mu). \tag{19}$$

The quantities η and ϵ are defined for needles and links as follows,

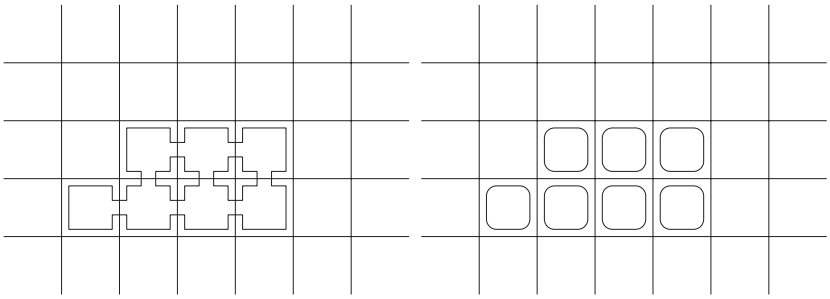
$$\eta(w) = \xi/(2a)^2 \tag{20}$$

$$\epsilon(l) = \begin{cases} 1 & \text{if} \quad l \text{ belongs to } 4k \text{ plaquettes,} \\ 1/2a & \text{if} \quad l \text{ belongs to } 4k+1 \text{ plaquettes,} \\ \xi & \text{if} \quad l \text{ belongs to } 4k+2 \text{ plaquettes,} \\ \xi/2a & \text{if} \quad l \text{ belongs to } 4k+3 \text{ plaquettes,} \end{cases} \tag{21}$$

where $\xi = +1$ for links and needles in direction 1, 2, 3 and $\xi = -1$ for links in direction 0.

We must also include the sign $(-1)^{L(C)}$ as an intrinsic feature of plaquettes. We achieve this in the following way:

(1) Let us take a site x having at least one adjacent plaquette. There are only 24 squares connected to the same site but the number of plaquettes put on these squares is arbitrary.
(2) Each plaquette adjacent to the site x is described by four numbers $(\mu, \nu, \alpha, \beta)$, where μ is the direction into the site x and ν is the direction out of x. The sign α is equal to ± 1 if the site $x \pm a_\mu$ belongs to the plaquette and β is defined analogously for ν.
(3) Let us consider one edge connected to x denoted by (μ, α), where μ is the direction of that edge and $\alpha = \pm 1$ same as for plaquettes. There could be plaquettes going through the edge (μ, α) into or out of x.
(4) Let us order separately all possible plaquettes having the same edge (μ, α) and going into and out of x ($(\mu, \nu, \alpha, \beta)$ and $(\nu, \mu, \beta, \alpha)$, respectively). The rule of ordering is the following: we first put all plaquettes with $b_\nu = -1$, then with $b_\nu = +1$ and order them according to increasing ν. The order of plaquettes within the same square is arbitrary but the same for every edge of that square. We obtain the function $n(\mu, \alpha, p)$ by assigning subsequent numbers $1, 2, 3, \ldots$, to the plaquettes p according to the above ordering.
(5) Now, we make the permutation σ on the set of plaquettes adjacent to x in the following way: let us take all plaquettes p going out of x and q into x though the edge (μ, α). Now, $\sigma(p) = q$ if $n(\mu, \alpha, p) = n(\mu, \alpha, q)$.
(6) It is possible that one plaquette p in the domain of σ is left. Then also one q in the image is left. Then we just put $\sigma(p) = q$. This means that C passes x.



(*a*) Site loops and the main loop (*b*) Plaquette loops

Figure 3. Switching of loops. It works in all dimensions but the picture would be obscure.

(7) The sign for the site is given by

$$\chi(x) = \mathrm{sgn}(\sigma)(-1)^{k(x)+\epsilon_0} \tag{22}$$

where $k(x)$ is the number of plaquettes adjacent to x. The number ϵ_0 is equal to 1 if the operation in step (6) is performed and 0 otherwise.

To show that we get an appropriate sign for a loop using only the internal structure of the surface, we find a relation between the product of signs of every site and the number of plaquettes. The sign constructed in (22) is just the number of loops (cycles) around each site. We can obtain simple loops around each plaquette using the loops at sites and the main loop surrounding the surface by switching links on edges (figure 3), The procedure is independent of the number of dimensions and plaquettes.

Now, we arrive at

$$Z_C = \prod_x \chi(x) m^{\zeta(x)} \prod_l \bar{\epsilon}(l) \prod_w \bar{\eta}(w)/m^2 \prod_p \exp\left[\mathrm{i}eF_{\mu\nu}(p)\right] \epsilon_t(p) \tag{23}$$

where we perform the product over sites x, links l, needles w and plaquettes p. The functions $\bar{\epsilon}$ and $\bar{\eta}$ have the same definitions as in (20) and (21) but with $\bar{\xi} = -\xi$. The sign $\epsilon_t(p)$ is equal to +1 for 12, 23 and 31, and -1 otherwise. The function $\zeta(x)$ is equal to 1 if the operation in step (6) is performed in x and 0 otherwise. Needles cannot touch one another and their ends must have $\zeta = 0$. The obtained Z_C corresponds to Q from equation (4). We leave the problem of the choice of the proper space of configurations for the next section. We have neglected the problem of fermion doubling which would force us to slightly alter the presented construction. We show this in appendix A.

4. The 1–1 correspondence

We would like to have the following relation between (8) and Z_C:

$$Z_0 \det(\mathrm{i}\not{D} - m) = \sum_{\{p,w\}} Z_C = Z \tag{24}$$

for a certain choice of configurations where Z_0 is a constant independent of the field A_μ.

It is impossible to establish 1–1 correspondence between fermionic determinant and bosonic partition function (24) using only a finite number of plaquettes per square (see appendix B).

We have however the 1–1 correspondence, though approximate, for an infinite number of plaquettes but we must include a damping term in $F_{\mu\nu}$, namely $F_{\mu\nu} \to F_{\mu\nu} + \mathrm{i}\delta$. The

sum of expressions (23) is now finite. For a lattice of size N we have $|Z| \leqslant (1 - \mathrm{e}^{-\delta})^{-8N^4}$. Allowed configurations are such that we cannot put two plaquettes with different orientations in the same place. Moreover, no link can have the number of ingoing and outgoing plaquettes differing by more than 1 and no point can have more than two adjacent edges with an odd number of plaquettes.

The number of plaquette configurations corresponding to a given loop configuration is nearly constant. To prove this, let us take the configuration without loops. The sum over configurations realizing such a structure is Z_0. Now let us put the least surface spanning a loop configuration C on every configuration of Z_0. If we have several such surfaces we choose one. This procedure is unique—no two different loop configurations have the same minimal surface. Every square s has originally m plaquettes (let m be integer with negative values for oppositely oriented plaquettes). The new configuration has either $m+1$, $m-1$ or m plaquettes if the surface goes through s with right or opposite orientation or omits s, respectively. Thus, we obtain the following bound for the sum of configurations giving C:

$$Z_0\,\mathrm{e}^{-S\delta} \leqslant \sum_{\{p(C),w(C)\}} Z_C \leqslant Z_0\,\mathrm{e}^{+S\delta} \tag{25}$$

where S is the number of plaquettes in the surface spanning C and the sum is restricted to plaquette configurations spanned on C. With $\delta \to 0$, we arrive at the expected result (24).

5. Discussion

We reformulated the fermionic determinant in terms of a bosonic partition function. The Pauli minus sign is local in this approach and the quantitative correspondence between bosonic and fermionic partition functions is obtained. By introducing many plaquettes, we obtained asymptotic equivalence to QED. The consequence is thrilling: one can imagine QED without Grassmann variables and anticommutation relations or, in simple words, without fermions.

The presented model looks very complicated, which was inevitable because we wanted to get an exact asymptotic equivalence to standard QED. It is useless for lattice numerical computation in the present form. But one should expect that the reality is much simpler, isotropic and continuous. Our model is only a strongly convincing proof that it is really possible. It is the author's current aim to construct such a better model.

Acknowledgment

The work was financially supported by The Stipend for Young Scientists awarded to the author by The Foundation for Polish Science.

Appendix A

One can easily find that there are 16 orthogonal solutions of the lattice equation

$$\mathrm{i}\,\partial\!\!\!/\psi = 0 \tag{A.1}$$

instead of 4. In order to cut off unwanted species one can write the action in the form

$$S = \int \mathrm{d}^4x\ \bar\psi\left(\mathrm{i}D\!\!\!/ - m + r\left(D_0^2 + D_1^2 + D_2^2 + D_3^2\right)\right)\psi(x), \tag{A.2}$$

where the Wilson term [9] $r\bar\psi\left(D_0^2 + D_1^2 + D_2^2 + D_3^2\right)\psi = \bar\psi Q\psi$ with $Q = Q_a + Q_b$ makes the three additional (Wilson) fermions that have very large masses. The discretized form of the

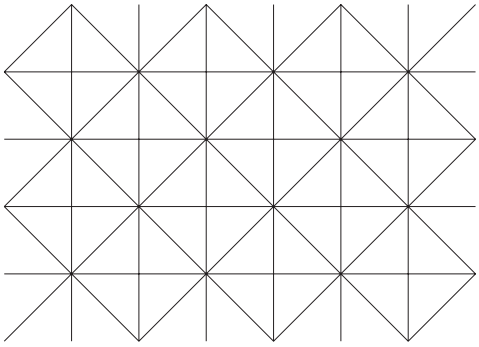


Figure 4. Triangularization in xy and zt planes (compare figures 1(a) and (b)).

operator Q on the lattice is as follows,

$$\begin{cases} Q_a(x,x) = -2b \\ Q_a(x, x \pm a_0 \pm a_3) = b\exp(\mathrm{i}ea(\pm A_0 \pm A_3)) \end{cases} \quad \text{for} \quad \begin{cases} n_0 + n_3 \equiv 0 \mod 2, & n_1 + n_2 \equiv 0, 1 \mod 4 \\ n_0 + n_3 \equiv 1 \mod 2, & n_1 + n_2 \equiv 2, 3 \mod 4 \end{cases}$$
$$\begin{cases} Q_b(x,x) = -2b \\ Q_b(x, x \pm a_1 \pm a_2) = b\exp(\mathrm{i}ea(\pm A_1 \pm A_2)) \end{cases} \quad \text{for} \quad \begin{cases} n_1 + n_2 \equiv 0 \mod 2, & n_0 + n_3 \equiv 0, 1 \mod 4 \\ n_1 + n_2 \equiv 1 \mod 2, & n_0 + n_3 \equiv 2, 3 \mod 4 \end{cases} \tag{A.3}$$

where $b = r/(2a)^2 \gg m$. In this way, we must introduce diagonal links on the lattice in planes 03 and 12 with weights $\exp(\mathrm{i}ea(\pm A_0 \pm A_3))$ and $\exp(\mathrm{i}ea(\pm A_1 \pm A_2))$ (see figure 4). Accordingly, we must include the triangles in plaquette formulation

$$Z_C = \prod_x \chi(x) M^{\zeta(x)} \prod_l \bar{\epsilon}(l) \prod_w \bar{\eta}(w)/m^2 \prod_p \exp[\mathrm{i}eF_{\mu\nu}(p)]\epsilon_t(p), \tag{A.4}$$

where for triangular plaquettes in planes 30 and 12 we define

$$\epsilon_t(p) = \begin{cases} (-1)^{n_0+n_3} & \text{for plaquettes oriented clockwise in plane 30 (figure 1}(a)\text{)}, \\ 1 & \text{for plaquettes oriented counterclockwise in plane 30 (figure 1}(a)\text{)}, \\ \mathrm{i} & \text{for plaquettes oriented clockwise in plane 12 (figure 1}(b)\text{)}, \\ -\mathrm{i} & \text{for plaquettes oriented counterclockwise in plane 12 (figure 1}(b)\text{)}. \end{cases} \tag{A.5}$$

The number $n_0 + n_3$ refers to the end of the slanted side. The function $\bar{\epsilon}$ is defined for slanted edges by the relation

$$\bar{\epsilon}(l) = \begin{cases} 1 & \text{for the link passed } 4k+1 \text{ times}, \\ b & \text{for the link passed } 4k+1 \text{ times}, \\ -1 & \text{for the link passed } 4k+2 \text{ times}, \\ -b & \text{for the link passed } 4k+3 \text{ times}. \end{cases} \tag{A.6}$$

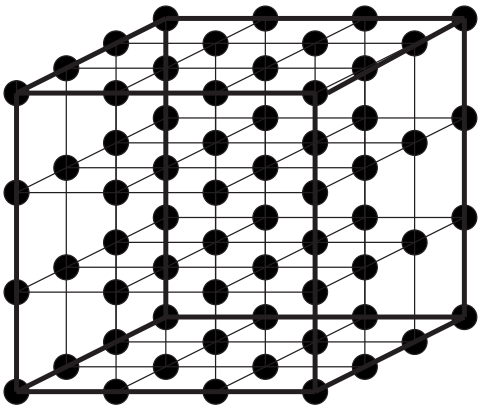


Figure 5. The cube of electrons—a counterexample to a finite number of plaquettes per square.

The function M is defined as follows:

$$M = \begin{cases} m & \text{for } (n_0+n_3, n_1+n_2) \equiv (0,3), (1,1), (2,2), (3,0) \quad \mod 4, \\ m+2b & \text{for } (n_0+n_3, n_1+n_2) \equiv (0,1)(1,0)(2,3)(3,2)(1,3) \\ & \qquad \times\ (3,1)(0,2)(2,0) \quad \mod 4, \\ m+4b & \text{for } (n_0+n_3, n_1+n_2) \equiv (0,0), (1,2), (2,1), (3,3) \quad \mod 4. \end{cases} \tag{A.7}$$

Appendix B

Let fermionic loops be arranged in a tight cubic net of size N in space (figure 5) and straight along the time direction. One can imagine a lattice of static electrons. Then every loop (electron) defines an edge of a plaquette surface. At every time slice, the plaquettes are seen as chains of links starting at every electron. The total number of chains beginning inside the cube is N^3, but the size of its surface is only $6N^2$. So there must be a link in space (square in spacetime) passed many times.

References

[1] Weinberg S 1996 *The Quantum Theory of Fields* (Cambridge: Cambridge University Press)
[2] Wilson K G 1974 *Phys. Rev.* D **10** 2445
[3] Kogut J B 1979 *Rev. Mod. Phys.* **51** 659
[4] Kogut J and Susskind L 1975 *Phys. Rev.* D **11** 395
[5] Susskind L 1977 *Phys. Rev.* D **16** 3031
[6] Fort H and Gambini R 1991 *Phys. Rev.* D **44** 1257
[7] Aroca J M, Fort H and Gambini R 1998 *Phys. Rev.* D **57** 3701
[8] Wen X-G 2003 *Phys. Rev.* D **68** 065003
[9] Wilson K G 1977 *New Phenomena in Subnuclear Physics, Proc. 14th School of Subnuclear Physics (Erice, 1975)* ed A Zichichi (New York: Plenum)
[10] Karsten L and Smit J 1981 *Nucl. Phys.* B **183** 103